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APPROXIMATION BY MEAN OF THE FUNCTION GIVEN BY DIRICHLET SERIES BY ABSOLUTELY CONVERGENT DIRICHLET SERIES

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Abstract

It is proved an uniform on compact sets approximation by mean of the general Dirichlet series.

Let $s = \sigma + it$ be a complex variable, $\{\lambda_m,\ m \in \mathbb{N}\}$ be an increasing sequence of real numbers such that $\lim_{m\to\infty} \lambda_m = +\infty$, and let $\{a_m,\ m \in \mathbb{N}\}$ be a sequence of complex numbers ($\mathbb{N}$ denotes the set of all natural numbers). The series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \tag{1}$$

is called a Dirichlet series with coefficients a_m and exponents λ_m. It is well known that the region of the convergence as well as of the absolute convergence of Dirichlet series is a half-plane. Suppose that the series (1) converges absolutely, for $\sigma > \sigma_a$ and denote its sum by $f(s)$. Then we have that $f(s)$ is a regular function on the half-plane $\sigma > \sigma_a$.

Suppose that the function $f(s)$ is analytically continuable to the region $\sigma > \sigma_a - \sigma_0$, where $\sigma_0 > 0$. Denote by B a number (not always the same) bounded by a constant. Let, for $\sigma > \sigma_a - \sigma_0$,

$$f(s) = B|t|^a, \qquad |t| \geq t_0, \tag{2}$$

with a certain constant $a > 0$, and

$$\int_0^T |f(\sigma + it)|^2\, dt = BT, \qquad T \to \infty. \tag{3}$$

In the theory of Dirichlet series an approximation by mean of the function $f(s)$ by absolutely convergent Dirichlet series plays an important role. This is done, see, for example, [1], for ordinary Dirichlet series for which $\lambda_m = \log m$. The aim of this note is to obtain a result of a such kind for general Dirichlet series (1).

Let $\sigma_1 > \sigma_0$. We define a function

$$l_n(s) = \frac{s}{\sigma_1}\,\Gamma\Big(\frac{s}{\sigma_1}\Big)e^{\lambda_n s}, \qquad \sigma \in [-\sigma_1, \sigma_1].$$

Here, as usual, $\Gamma(s)$ denotes the Euler gamma-function. We will consider, for $\sigma > \sigma_a - \sigma_0$, the following function

$$f_n(s) = \frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} f(s+z) l_n(z)\,\frac{dz}{z}\,.$$

In view of the equality

$$|\Gamma(s)| = \sqrt{2\pi}\,|t|^{\sigma - 1/2}e^{-\pi|t|/2}\big(1 + B|t|^{-1}\big), \qquad |t| \geq t_0,$$

and of the condition (2) we have that the integral for $f_n(s)$ exists.

Lemma. *We have*

$$f_n(s) = \sum_{m=1}^{\infty} a_m \exp\big\{-e^{(\lambda_m - \lambda_n)\sigma_1}\big\}e^{-\lambda_m s}\,,$$

the series being absolutely convergent for $\sigma > \sigma_a - \sigma_0$.

Proof. Since $\sigma_1 > \sigma_0$, we see that $\sigma + \sigma_1 > \sigma_a$. Hence the function $f(s+z)$ for $\operatorname{Re} z = \sigma_1$ can be presented by the absolutely convergent Dirichlet series

$$f(s+z) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m(s+z)}.$$

Let

$$b_n(m) = \frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} l_n(s) e^{-\lambda_m s}\,\frac{ds}{s}\,,$$

and consider the series

$$\sum_{m=1}^{\infty} a_m b_n(m) e^{-\lambda_m s}. \tag{4}$$

In view of the estimate

$$b_n(m) = Be^{-\lambda_m \sigma_1}\int\limits_{-\infty}^{\infty} |l_n(\sigma_1 + it)|\,dt = Be^{-\lambda_m \sigma_1}$$

the series (4) absolutely converges for $\sigma > \sigma_a - \sigma_0$. Therefore we may change sum and integral in the definition of $f_n(s)$. This gives

$$f_n(s) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} l_n(z) e^{-\lambda_n z} \frac{dz}{z} = \sum_{m=1}^{\infty} a_m b_n(m) e^{-\lambda_m s}. \tag{5}$$

For positive b and c the following formula

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) b^{-s}\, ds = e^{-b}$$

is true [2]. Consequently,

$$b_n(m) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{s}{\sigma_1} \Gamma\Big(\frac{s}{\sigma_1}\Big) e^{-(\lambda_m - \lambda_n)s} \frac{ds}{s} =$$

$$\frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \Gamma\Big(\frac{s}{\sigma_1}\Big) e^{(\lambda_m - \lambda_n)(-s/\sigma_1)\sigma_1}\, d\Big(\frac{s}{\sigma_1}\Big) =$$

$$\exp\Big\{ - e^{(\lambda_m - \lambda_n)\sigma_1} \Big\}.$$

This together with (5) proves the lemma.

Denote by D the half-plane $\sigma > \sigma_a - \sigma_0$.

Theorem. *Let K be a compact subset of D. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \big| f(s+i\tau) - f_n(s+i\tau) \big|\, d\tau = 0.$$

Proof. We begin with the change of the contour of integration in the definition of $f_n(s)$. Clearly, the integrand in the definition of $f_n(s)$ has a simple pole at the point $z = 0$. Let $\varepsilon > 0$ and $\sigma_1 > 0$ be such that σ belongs to $[\sigma_a - \sigma_0 + \varepsilon, \sigma_1]$ when $s \in K$. We take

$$\sigma_2 = \sigma_a - \sigma_0 + \frac{\varepsilon}{2}.$$

Then the residue theorem yields for $\sigma \in [\sigma_a - \sigma_0 + \varepsilon, \sigma_1]$

$$f_n(s) = \frac{1}{2\pi i} \int_{\sigma_2 - \sigma - i\infty}^{\sigma_2 - \sigma + i\infty} f(s+z) l_n(z) \frac{dz}{z} + f(s). \tag{6}$$

Let L be a simple closed contour lying in D and enclosing the set K, and let δ denote the distance of L from the set K. Then by the Cauchy formula

$$f(s+i\tau)-f_n(s+i\tau)=\frac{1}{2\pi i}\int\limits_L \frac{f(z+i\tau)-f_n(z+i\tau)\,dz}{z-s},$$

where $s\in K$, we have

$$\sup_{s\in K}\bigl|f(s+i\tau)-f_n(s+i\tau)\bigr|\le\frac{1}{2\pi\delta}\int\limits_L \bigl|f(z+i\tau)-f_n(z+i\tau)\bigr|\,|dz|.$$

Therefore, for sufficiently large T, we obtain

$$\frac{1}{T}\int\limits_0^T \sup_{s\in K}\bigl|f(s+i\tau)-f_n(s+i\tau)\bigr|\,d\tau=$$

$$\frac{B}{T\delta}\int\limits_L |dz|\int\limits_0^{2T}\bigl|f(\operatorname{Re}z+i\tau)-f_n(\operatorname{Re}z+i\tau)\bigr|\,d\tau+\frac{B|L|}{T\delta}= \tag{7}$$

$$\frac{B|L|}{T\delta}+\frac{B|L|}{T\delta}\sup_{s\in L}\int\limits_o^{2T}\bigl|f(\sigma+it)-f_n(\sigma+it)\bigr|\,dt.$$

Here $|L|$ is the lenght of the contour L. Now we choose the contour L so that, for $s\in L$,

$$\sigma\ge\sigma_a-\sigma_0+\frac{3\varepsilon}{4},\qquad \delta\ge\frac{\varepsilon}{4}.$$

The formula (6) for such σ yields

$$f(\sigma+it)-f_n(\sigma+it)=B\int\limits_{-\infty}^{\infty}\bigl|f(\sigma_2+it+i\tau)\bigr|\bigl|l_n(\sigma_2-\sigma+i\tau)\bigr|\,d\tau.$$

Hence, for the same σ, we find that

$$\frac{1}{T}\int\limits_0^{2T}\bigl|f(\sigma+it)-f_n(\sigma+it)\bigr|\,dt=$$

$$B\int\limits_{-\infty}^{\infty}\bigl|l_n(\sigma_2-\sigma+i\tau)\bigr|\,\frac{1}{T}\int\limits_{-|\tau|}^{|\tau|+2T}\bigl|f(\sigma_2+it)\bigr|\,dt\,d\tau. \tag{8}$$

Taking into account the estimate (3), we obtain that

$$\int\limits_{-|\tau|}^{|\tau|+2T} \big|f_n(\sigma_n+it)\big|\,dt \le \left(\int\limits_{-|\tau|}^{|\tau|+2T} \big|f_2(\sigma_2+it)\big|^2\,dt\right)^{1/2} \big(2T+2|\tau|\big)^{1/2} = B\big(2T+2|\tau|\big).$$

Thus, (8) implies the estimate

$$\begin{aligned}
&\frac{1}{T}\sup_{\substack{\sigma\\ s\in L}} \int\limits_0^{2T} \big|f(\sigma+it)-f_n(\sigma+it)\big|\,dt = \\
&B\sup_{\substack{\sigma\\ s\in L}} \int\limits_{-\infty}^{\infty} \big|l_n(\sigma_2-\sigma+it)\big|\left(1+\frac{|t|}{T}\right)dt = \qquad (9)\\
&B\sup_{\sigma\in[-\sigma_1,-\varepsilon/4]} \int\limits_{-\infty}^{\infty} \big|l_n(\sigma+it)\big|(1+|t|)\,dt.
\end{aligned}$$

However, the definition of $l_n(s)$ gives

$$\lim_{n\to\infty}\sup_{\sigma\in[-\sigma_1,-\varepsilon/4]} \int\limits_{-\infty}^{\infty} |l_n(\sigma+it)\big|(1+|t|)\,dt = 0.$$

This, (7) and (8) completes the proof of the theorem.

REFERENCES

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2. E. C. Titchmarsh, *The Theory of Functions,* (in Russian), Nauka, Moscow, 1980.